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Writing Sample: Excerpt of Report, 4/11/2018

The following material is an excerpt from a recently completed non-published contract report completed on April 11, 2018. The mathematical formulation presented below was then used in the implementation of a new interface element for the finite element codes Abaqus/Standard and Abaqus/Explicit through the User-Element Subroutine (UEL). Additional writing samples are available upon request.

--------- *Excerpt begins* ---------

1 Mathematical Formulation: Interior Penalty DGM

Overview

Discontinuous Galerkin (DG) methods are versatile numerical techniques in computational mechanics that generalize finite element weak formulations by allowing jump-discontinuities in the problem unknowns across interior boundaries [1]. In contrast to continuous Galerkin (CG) methods, each discontinuous subdomain is permitted to use independent element shape functions, non-conforming meshes, and even distinct physical models [2]. Each region is then coupled to its neighbors through boundary integrals which naturally occur when integrating by parts over each subdomain [1].

The Interior Penalty DGM (IP DGM) is a specific DG method that guarantees the consistency and stability of the solution based on the selection of appropriate numerical "fluxes" across interelement boundaries. Although many authors have contributed to the method over the years, the origin of the IP DGM is usually traced to Nitsche [3] where it was used to enforce Dirichlet boundary conditions. Indeed, the general formulation is still often referred to as "Nitsche's method" in the literature. For authoritative general references of DG methods, we point the reader first to the works published by Arnold et al. [4], and by Cockburn [5]. For the specific application of the IP DGM for solid mechanics, we have found the works by Hansbo and Hansbo in [6] and [2], by Kaufmann[7], and by Noels and Radovitzky [1] to be both useful and concise.

For our purposes, the main advantages provided by DG methods include:

- Elimination of artificial compliance across interelement boundaries prior to the onset of damage.

- Importantly, the IP DGM is suitable for both explicit and implicit solvers[1].
- A powerful way to represent discontinuities present in a physical model.

Like the strategy used in the MPC-based version of the selectively activated CZM, to achieve our goals we require the ability to:

- Activate/deactivate the IP formulation at any interelement boundary to model the growth of cracks.
- Base the activation criteria on the stress state within the elements on either side of an interface.
- Use the technique with Abaqus.

The IP DGM is not available in any commercially available FEM code. Therefore, the implementation developed for this report is based on the references cited above and has been programmed through various user subroutines so that the approach is compatible with Abaqus.

General Form of the IP DGM

The IP DGM enforces continuity of the solution across discontinuous elements through additional terms in the finite element weak form. As we show in the sections below, the general form is obtained by integrating by parts and summing contributions over each element K and at each interior interface Γ in the mesh, to arrive at:

$$\begin{aligned}&\sum_{K}\int_{K}\varepsilon(v):C:\varepsilon(u)\\&-\sum_{\Gamma\in\Gamma_I}\left(\int_{\Gamma}[\![v]\!]:\{\sigma(u)\}+\int_{\Gamma}[\![u]\!]:\{\sigma(v)\}-\int_{\Gamma}\eta[\![v]\!]:[\![u]\!]\right)=\sum_{K}\int_{K}f\cdot v\end{aligned} \tag{1.1}$$

where:

[1] **Implementation Note:** The IP DGM is well-suited for parallelization in the explicit FEM.

$u \equiv vector\ of\ nodal\ displacements$
$v \equiv vector\ test\ function\ (virtual\ displacements)$
$f \equiv vector\ of\ forces$
$C \equiv material\ constitutive\ tensor$
$\eta \equiv penalty\ parameter$
$\varepsilon(v) \equiv \nabla v \equiv tensor\ test\ function\ (virtual\ strains)$
$\varepsilon(u) \equiv \nabla u \equiv strain\ tensor$
$\sigma(v) \equiv \left(C : \varepsilon(v)\right) \equiv Cauchy\ stress\ tensor$
$\sigma(u) \equiv \left(C : \varepsilon(u)\right) \equiv Cauchy\ stress\ tensor$

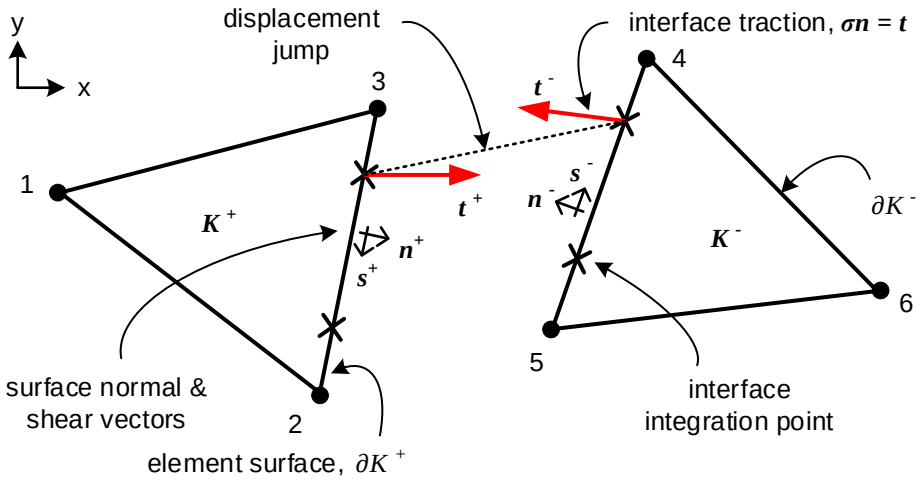

Figure 1-1. Two adjacent elements in the Interior Penalty method, where the interface is defined as $\Gamma = \cup\, \partial K^{\pm}$ and numerical fluxes are based on the jump and average of element stresses and displacements.

Referring to Figure 1-1, let the interface between two adjacent, but discontinuous, elements $K^{\pm}$ be represented by Γ, where $\Gamma \cup \partial K^{\pm}$. For convenience, we will denote the "+" or "-" sides of an interface as "1" and "2". Note that the order in which the elements are given is not important in our formulation, but once chosen the sides are fixed.

The average operator $\{\bullet\}$ and the jump operator $[\![\bullet]\!]$ are used in (1.1) to describe the limit values of stresses and displacements at each interface, Γ. The average operator simply computes the *average* of the matrix-valued/tensor quantity (σ) at an interface:

$$\{\sigma\} = \frac{1}{2}(\sigma_1 + \sigma_2)$$

The jump operator describes the *difference* in a vector quantity (v) at an interface[2]:

$$[\![\mathrm{v}]\!] = \mathrm{v}_1 \otimes n_1 + \mathrm{v}_2 \otimes n_2$$

The first and third terms in (1.1) are the same as those found in the conventional (continuous Galerkin) FEM weak form to compute the work due to internal element stresses and the work due to external loads, respectively. The IP DGM includes three additional integrals for each interior interface Γ to account for the work done across adjacent element surfaces, with the following important properties (c.f. [4, 5, 7-9]):

- The first interface integral in (1.1) is called the *consistency* term. This term guarantees that the IP method will find any continuous solution u that can be represented by the finite element shape functions, such that $[\![u]\!] = 0$.

- The second interface integral is called the *symmetry* term. When combined with the consistency term the IP method becomes symmetric in u and v. Accordingly, the resulting interface stiffness matrices are also symmetric.

- The third interface integral is called the *stability* term. This term acts to weakly enforce displacement compatibility and the stability of the IP method. The stability term has a form like that used for the cohesive penalty-based TSL. However, a lower bound for the penalty factor η can be found that stabilizes the method and guarantees that the interface stiffness matrix is positive-definite.

Derivation of the IP DGM

The following derivations are based on those found in [7],[4] with some departures which affect the final form of the IP DGM equations and the overall presentation.

1.1.1 Discontinuous Discretization

In the Discontinuous Galerkin Method (DGM), we refer to the subdivision of a domain Ω into multiple non-overlapping subregions separated by internal discontinuities. To help formalize the definitions used in the following sections, a simple example is shown in Figure 1-2, where:

[2] **Implementation Note:** Both v and n are column-vectors, so that with the outer product defined as $(\mathrm{v} \otimes n) = (\mathrm{v} \cdot n^T)$ the jump operator results in a matrix-valued output.

$$\Omega \equiv Complete\ Domain$$
$$\partial\Omega \equiv Boundary\ Surface\ of\ \Omega$$
$$\Omega_1 \equiv Subregion\ 1$$
$$\Omega_2 \equiv Subregion\ 2$$
$$\partial\Omega_i \equiv Boundary\ Surfaces\ of\ i^{th}\ Subregion$$
$$\Gamma \equiv Internal\ Discontinuity$$

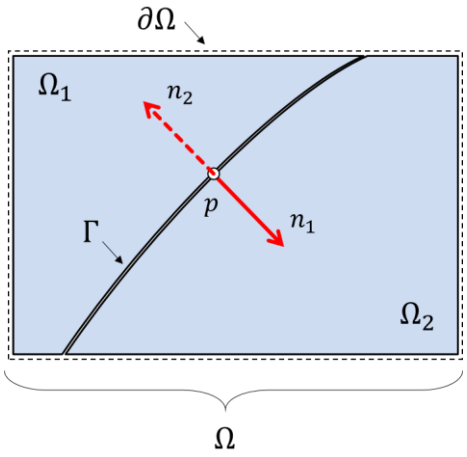

Figure 1-2. Example subdivision of a domain.

The complete domain Ω is defined as the collection of all E subregions, $\Omega = \cup_i^E \Omega_i$. Let $\partial\Omega$ be the (external) boundary of Ω. We assume that each subregion is non-overlapping, such that $\Omega_i = \Omega \backslash \Omega_{j,j\neq i}$, and the boundary of the i^{th} subregion is represented by the set $\partial\Omega_i$. Let Γ be the intersection of two adjacent subregions, $\Gamma = \Omega_i \cap \Omega_{j,j\neq i}$. Then, each point p on Γ may be split into corresponding points $p^{\pm}$ from the surfaces $\partial\Omega^{\pm}$ with normal vectors $n^{\pm}$. Then, for example, in Figure 1-2 we have:

$$E = 2$$
$$\Omega = \cup_i^2 \Omega_i = \Omega_1 \cup \Omega_2$$
$$\Omega_1 = \Omega \backslash \Omega_2$$
$$\Omega_2 = \Omega \backslash \Omega_1$$
$$\Gamma = \Omega_1 \cap \Omega_2$$

Discretization of the domain into elements for the FEM results in an approximation of Ω represented by Ω^h. We may then view each subdivision of Ω^h as an element K with faces ∂K. We next define $\Gamma_* = \cup^E \partial K$ as the set that collects all internal and external element faces over the complete domain.

We assume that discontinuity surfaces occur only at interior interelement boundaries, aka the interfaces. Therefore, in a slight abuse of notation, we reuse the symbol Γ to represent an interface within the discretized domain. Thus, an interior interface Γ represents the set of all points at the shared boundary of two adjacent elements, so that $\Gamma = K^+ \cap K^- = \partial K^+ \cup$

∂K^-, and $\Gamma \subset \partial K^\pm$. Let F be the total number of interior interface sets. We then let the set of all F interior interface sets be represented by $\Gamma_I = \cup^F \Gamma$.

We call each generic surface $f \in \Gamma_*$ of the discretized domain a "mesh face". On the external boundary, a mesh face is the exterior element face $f = \partial K \in \partial\Omega^h$. On the interior of the mesh, a mesh face is the union of the faces for two adjacent elements, so that $f = \Gamma \in \Gamma_I$.

Remark: In a fully discontinuous mesh, each element contains a unique set of DOF that are not shared with any other element, and the IP DGM formulation is used at every interior interface. However, in the present work we permit meshes to be split at all interelement boundaries, or at specified locations only. Therefore, the IP DGM may be used at any subset of interelement boundaries, while the standard continuous FEM is used elsewhere. This flexible implementation is useful for minimizing the introduction of additional DOF wherever they are not desired.

1.1.2 Limit Values

The average and jump in interface quantities (such as stress σ and displacements u) are the limit values taken from $K^\pm$, evaluated at corresponding points $p^\pm$ on element boundaries $\partial K^\pm$ with surface normals $n^\pm$ which are originally coincident when an interface is closed.

1.1.3 Average Operator

The average operator $\{\bullet\}$ simply computes the average of a vector quantity (v) or of a matrix/tensor quantity (σ) at an interface:

$$\{\mathrm{v}\} = \frac{1}{2}(\mathrm{v}_1 + \mathrm{v}_2) \tag{1.2}$$

$$\{\sigma\} = \frac{1}{2}(\sigma_1 + \sigma_2) \tag{1.3}$$

1.1.4 Jump Operator

The jump operator $[\![\bullet]\!]$ describes the difference in a vector quantity (v) or a matrix/tensor quantity (σ) at an interface:

$$[\![\mathrm{v}]\!] = \mathrm{v}_1 \otimes n_1 + \mathrm{v}_2 \otimes n_2 \tag{1.4}$$

$$[\![\sigma]\!] = \sigma_1 \cdot n_1 + \sigma_2 \cdot n_2 \tag{1.5}$$

Note that in (1.4), v and n are column-vectors, and $\mathrm{v} \otimes n = \mathrm{v} \cdot n^T$ is the outer product. Also, note that $[\![\bullet]\!]$ maps vectors to matrices, and matrices to vectors in the direction of the surface normal n.

1.1.5 Average and Jump on an External Boundary

The average and jump of the displacement vector, u, for a point on an element face on the external boundary, $\partial K \in \partial \Omega^h$, is defined as:

$$\{u\} = u_1 = u \tag{1.6}$$

$$[\![u]\!] = u_1 \otimes n_1 = u \cdot n^T \tag{1.7}$$

1.1.6 Scalar Jump Operator

Following the notation of Kaufmann [7], we also define a scalar jump operator, which maps scalars to vectors:

$$\begin{aligned} [\![c]\!]_s &= c_1 \cdot n_1 + c_2 \cdot n_2 \\ &= \mathrm{v} \end{aligned} \tag{1.8}$$

and maps vectors to scalars:

$$\begin{aligned} [\![\mathrm{v}]\!]_s &= \mathrm{v}_1 \cdot n_1 + \mathrm{v}_2 \cdot n_2 \\ &= c \end{aligned} \tag{1.9}$$

1.1.7 Jump-Average Identity

The jump operator, average operator, and scalar jump operator are related by the identity:

$$[\![\mathrm{v} \cdot \sigma]\!]_s = [\![\mathrm{v}]\!] : \{\sigma\} + \{\mathrm{v}\} \cdot [\![\sigma]\!] \tag{1.10}$$

Note that the identity maps a vector $(\mathrm{v} \cdot \sigma)$ to a scalar, as in (1.9). The proof of the identity is given in Appendix A.

1.1.8 Ordering

The interface equations developed for this report are based on an “ordering independent” formulation. Contributions from either side of an interface are added together to generate the IP constraint, so that whether a side is chosen as “+” or “-” is completely arbitrary. Once chosen, however, the specified sides are fixed so that contributions from each side are properly assembled to the global system of equations.

In the current work, a discontinuous mesh is constructed by visiting each element and splitting it from its neighbors. For each element-neighbor pair, the current element being visited is assigned to side “1”, and its neighbor is assigned to side “2” (see Figure 1-1). To remain consistent with this scheme, in the following sections we choose to represent the adjacent elements with “1” and “2” instead of “+” and “-”, respectively. Of course, an element may be designated as either “1” or “2” for any element-neighbor pair it is involved in.

Remark: In the literature (c.f. [10],[11],[12]), it is common to see the jump operator defined with one of the alternatives:

$$[\![\bullet]\!] = (\bullet_1 - \bullet_2) \cdot n \quad \text{OR} \quad [\![\bullet]\!] = (\bullet_2 - \bullet_1) \cdot n$$

In either of the alternative forms above, a dependency on the order in which the elements are assigned is introduced. This is directly caused by the selection of a single normal vector n to describe the orientation of the interface. For example, selecting $-n_2 = n_1 = n$, we reach the expression $[\![\bullet]\!] = (\bullet_1 - \bullet_2) \cdot n$, so that values computed on side 2 of an interface are subtracted from the values computed on side 1. The benefit of using one of these “ordering-dependent” forms is that it clearly computes the *difference* in a quantity across an interface.

In contrast, the form shown in (1.4) or (1.5) is “ordering-independent”. This was also pointed out by Arnold, et al. [4], who stated, *“The advantage of these definitions is that they do not depend on assigning an ordering to the elements, K_i”*. In practice, we feel this distinction is relatively minor.

More importantly, the expression $[\![\bullet]\!] = (\bullet_1 - \bullet_2) \cdot n$ implies that the normal vectors on each side of the interface are colinear, such as illustrated in Figure 1-2. This simplification becomes embedded in the interface equations and is therefore assumed to hold true *at all times* during an analysis. This assumption is valid when the strength of the DG constraint is sufficient to keep the interface closed. On the other hand, in (1.4) and (1.5) the two normal vectors n_1 and n_2 are *not* assumed to be colinear.

In this report we apply the “ordering-independent” form of the jump operator. We then continue to preserve the distinction between the two surface normals n_1 and n_2 throughout all derivations and into the final discretized equations and numerical implementation. As we shall see, this choice manifests itself in the final form of the IP interface equations used to compute the balance of forces and enforce compatibility between adjacent elements. As such we consider it to be a more precise representation of the governing interface equations.

We do note that when the interface separation gap is small the surface normals are indeed approximately colinear. Mathematically, this condition is expressed as:

$$[\![u]\!] \approx 0 \quad \Longrightarrow \quad -n_2 \approx n_1 \tag{1.11}$$

Thus, the order-independent and order-dependent forms are functionally equivalent if the interface gap is small. Nevertheless, the distinction is important for activated interface elements in a cohesive zone, where larger separations occur during the damage process.

1.1.9 Strong and Weak Forms

From linear elasticity and continuum mechanics, we use the constitutive relation (1.12) and the strong form of the static equilibrium statement (1.13):

$$\sigma = C : \varepsilon(u) \tag{1.12}$$

$$\nabla \cdot \sigma = -f \tag{1.13}$$

Now, following the usual Galerkin method, we discretize the domain and multiply (1.12) by a tensor test function τ, and (1.13) by a vector test function v. Integrating over an individual element K, we obtain the weak form of the equations:

$$\int_K \tau : \sigma = \int_K \tau : C : \varepsilon(u) \tag{1.14}$$

$$\int_K \nabla \cdot \sigma \cdot v = -\int_K f \cdot v \tag{1.15}$$

In a discontinuous mesh, the DOF for each element K are independent and uncoupled from those of its neighbors. Therefore, we seek a method to control interactions between elements. We begin by expressing the constitutive equation (1.14) and equilibrium statement (1.15) as the sum of volume and surface contributions.

Integrating by parts over the right-hand side of (1.14), the constitutive equation may be rewritten the as:

$$\int_K \tau : \sigma = -\int_K \nabla \cdot (\tau : C) \cdot u + \int_{\partial K} u \cdot (\tau : C) \cdot n \tag{1.16}$$

Then, by applying the divergence theorem to the left-hand side of the weak equilibrium statement of (1.15) and rearranging, we obtain:

$$\int_K \sigma : \varepsilon(v) = \int_K f \cdot v + \int_{\partial K} v \cdot \sigma \cdot n \tag{1.17}$$

1.1.10 Flux Formulation

Both displacements u and stresses σ are dual-valued on the interelement boundaries $\partial K^{\pm}$ within a discontinuous mesh. Therefore, in the IP DGM we take contributions from each of the adjacent elements to define the numerical fluxes $\hat{u}$ and $\hat{\sigma}$:

$$\hat{u} = \{u\} \tag{1.18}$$

$$\hat{\sigma} = \{C : \varepsilon(u)\} - \eta[\![u]\!] \tag{1.19}$$

The displacement flux $\hat{u}$ is a vector that describes the average displacement for corresponding points $p^{\pm}$ across an interface. The stress flux $\hat{\sigma}$ describes the average stress across the interface, along with an additional term, $\eta[\![u]\!]$, that penalizes displacement jumps.

We seek to use these fluxes to enforce displacement continuity and traction equilibrium. Therefore, (1.16) and (1.17) may be rewritten in the so-called DG "flux formulation":

$$\begin{aligned} \int_K \tau : \sigma &= -\int_K \nabla \cdot (\tau : C) \cdot u + \int_{\partial K} \hat{u} \cdot (\tau : C) \cdot n \\ &= -\int_K \nabla \cdot (\tau : C) \cdot u + \int_{\partial K} \{\mathrm{u}\} \cdot (\tau : C) \cdot n \end{aligned} \tag{1.20}$$

$$\begin{aligned} \int_K \sigma : \varepsilon(v) &= \int_K f \cdot v + \int_{\partial K} v \cdot \hat{\sigma} \cdot n \\ &= \int_K f \cdot v + \int_{\partial K} v \cdot (\{C : \varepsilon(u)\} - \eta[\![u]\!]) \cdot n \end{aligned} \tag{1.21}$$

We focus first on the weak constitutive relation in (1.20). Summing over all elements in the mesh, we rewrite this equation as:

$$\sum_K \int_K \tau : \sigma = -\sum_K \int_K \nabla \cdot (\tau : C) \cdot u + \sum_K \int_{\partial K} \{u\} \cdot (\tau : C) \cdot n \tag{1.22}$$

Now, using the Divergence Theorem to rewrite the first term on the RHS of (1.22), we obtain:

$$\begin{aligned}\sum_K \int_K \tau : \sigma = \sum_K \int_K (\tau : C) : \varepsilon(u) \\ -\sum_K \int_{\partial K} u \cdot (\tau : C) \cdot n \\ +\sum_K \int_{\partial K} \{u\} \cdot (\tau : C) \cdot n\end{aligned} \tag{1.23}$$

where we have also used $\nabla \cdot u = \varepsilon(u)$. Note that in (1.23) there are now two surface integrals for each surface on every element. Combining them, we reach an updated form:

$$\sum_K \int_K \tau : \sigma = \sum_K \int_K (\tau : C) : \varepsilon(u) + \sum_K \int_{\partial K} (\{u\} - u) \cdot (\tau : C) \cdot n \tag{1.24}$$

We next focus on the final term on the RHS of (1.24). We make the following observations:

- This expression describes the average displacement vector, $\{u\}$, multiplied by a traction vector resulting from the stress within the material, $(\tau : C) \cdot n$. Thus, the final value is a scalar.
- The integral is computed at every internal and external element surface, for each element in the mesh – that is, for every surface f in the set $\Gamma_* = \cup^E \partial K$. Therefore, each interior element surface is integrated twice (once from each element on either side of the interface).

In addition, we point out:

- The displacement vector, $\hat{u} = \{u\}$, is single-valued on each internal and external surface in the set Γ_*.
- The boundary tractions, $(\tau : C) \cdot n$, are dual-valued on each internal surface.

Therefore, we use the scalar jump operator defined in (1.4) to rewrite the expression in a form that requires only a single integration over interior boundaries:

$$\sum_K \int_K \tau : \sigma = \sum_K \int_K (\tau : C) : \varepsilon(u) + \sum_{f \in \Gamma_*} \int_f [\![(\{u\} - u) \cdot (\tau : C)]\!]_s \tag{1.25}$$

Note that with the displacement vector $\{u\}$ on the exterior boundary $\partial\Omega^h$ defined in (1.6), the integral over the set Γ_* accounts for both interior and exterior mesh faces. We also note that this term vanishes when the displacements of corresponding points $p^{\pm}$ on either side of an interface are equal:

$$u_1 = u_2 \quad \Leftrightarrow \quad \{u\} = u \tag{1.26}$$

In addition, whenever (1.26) holds, it implies $[\![u]\!] = 0$ and the two element surfaces are coincident. Then the outward normal vectors on either side of the discontinuity are colinear, but pointed in opposite directions so that they have the relationship:

$$n_2 = -n_1 \tag{1.27}$$

Thus, (1.25) is also a valid expression for the standard (continuous Galerkin) finite element method, because the surface integral is potentially non-zero only for meshes with interior discontinuities.

We now focus on the weak equilibrium statement in (1.21). Summing over all elements in the mesh, we rewrite this equation as:

$$\sum_K \int_K \sigma : \varepsilon(v) = \sum_K \int_K f \cdot v + \sum_K \int_{\partial K} v \cdot (\{C : \varepsilon(u)\} - \eta[\![u]\!]) \cdot n \tag{1.28}$$

We make the following observations for the surface integral in the final term on the RHS:

- The vector test function v is dual-valued on each internal surface.
- The stress flux, $\hat{\sigma} = (\{C : \varepsilon(u)\} - \eta[\![u]\!])$ is single-valued on each internal and external element surface in the mesh.
- The surface integral is computed at every internal and external element surface, for each element in the mesh, so that each interior element surface is integrated twice (once from each element on either side of the interface).

Therefore, we again use the scalar jump operator to rewrite the expression in a form that requires only a single integration over interior boundaries. Then, the global equilibrium statement becomes:

$$\sum_K \int_K \sigma : \varepsilon(v) = \sum_K \int_K f \cdot v + \sum_{f \in \Gamma_*} \int_f [\![v \cdot (\{C : \varepsilon(u)\} - \eta[\![u]\!])]\!]_s \tag{1.29}$$

Thus, with the updated constitutive relation (1.25) and equilibrium statement (1.29), the flux formulation of the IP DGM is complete.

1.1.11 Primal Formulation

We next rewrite the flux formulation in its "primal form" so that it is a function of u alone, suitable for use with the typical displacement-based FEM. Let τ be a function of the gradient of the vector test function:

$$\tau = \tau(\nabla v) = \varepsilon(v)$$

The two equations of the DG flux formulation can then be given succinctly as:

$$\begin{aligned} A\text{:} \quad & \sum_K \int_K \sigma : \varepsilon(v) = \sum_K \int_K \varepsilon(v) : C : \varepsilon(u) + \sum_{f \in \Gamma_*} \int_f [\![(\hat{u} - u) \cdot \big(C : \varepsilon(v)\big)]\!]_s \\ B\text{:} \quad & \sum_K \int_K \sigma : \varepsilon(v) = \sum_K \int_K f \cdot v + \sum_{f \in \Gamma_*} \int_f [\![v \cdot \hat{\sigma}]\!]_s \end{aligned} \tag{1.30}$$

By substituting the constitutive relation (1.30.A) into the equilibrium equation (1.30.B), we may eliminate the explicit dependence on σ:

$$\sum_K \int_K \varepsilon(v) : C : \varepsilon(u) + \sum_{f \in \Gamma_*} \int_f [\![(\hat{u} - u) \cdot \big(C : \varepsilon(v)\big) - v \cdot \hat{\sigma}]\!]_s = \sum_K \int_K f \cdot v \tag{1.31}$$

Using the average operator, the jump operator, and the jump-average identity, we may now rewrite the surface integral in the second term of (1.31) as:

$$\begin{aligned} & \int_f [\![(\hat{u} - u) \cdot \big(C : \varepsilon(v)\big) - v \cdot \hat{\sigma}]\!]_s \\ & \quad = \int_f [\![(\{u\} - u)\, \sigma(v) - v \cdot (\{\sigma(u)\} - \eta [\![u]\!])]\!]_s \\ & \quad = \int_f \Big([\![\{u\} \cdot \sigma(v)]\!]_s - [\![u \cdot \sigma(v)]\!]_s - [\![v \cdot \{\sigma(u)\}]\!]_s + \big[\!\big[v \cdot \eta [\![u]\!] \big]\!\big]_s \Big) \\ & \quad = \int_f (\text{part1} - \text{part2} - \text{part3} + \text{part4}) \end{aligned} \tag{1.32}$$

For convenience, we have also assumed the constitutive law can be enforced strongly, so that we may write $\sigma(v) = C : \varepsilon(v) = C : \nabla v$, and likewise, $\sigma(u) = C : \varepsilon(u) = C : \nabla u$.

Each part of (1.32) is evaluated using the jump-average identity in (1.10):

part1: $$\begin{aligned}[\![\{u\}\cdot\sigma(v)]\!]_s &= [\![\{u\}]\!] : \{\sigma(v)\} + \{\{u\}\}\cdot[\![\sigma(v)]\!] \\ &= 0 + \{u\}\cdot[\![\sigma(v)]\!]\end{aligned}$$

part2: $$-[\![u\cdot\sigma(v)]\!]_s = -[\![u]\!] : \{\sigma(v)\} - \{u\}\cdot[\![\sigma(v)]\!]$$

part3: $$\begin{aligned}-[\![v\cdot\{\sigma(u)\}]\!]_s &= -[\![v]\!] : \{\{\sigma(u)\}\} - \{v\}\cdot[\![\{\sigma(u)\}]\!] \\ &= -[\![v]\!] : \{\sigma(u)\} - 0\end{aligned}$$

part4: $$\begin{aligned}\left[\!\left[v\cdot\eta[\![u]\!]\right]\!\right]_s &= [\![v]\!] : \{\eta[\![u]\!]\} + \{v\}\cdot\left[\!\left[\eta[\![u]\!]\right]\!\right] \\ &= +\eta[\![v]\!] : [\![u]\!] + 0\end{aligned}$$

Thus, after summing all parts in (1.32) we are left with:

$$-[\![u]\!] : \{\sigma(v)\} - [\![v]\!] : \{\sigma(u)\} + \eta[\![v]\!] : [\![u]\!]$$

With this result we can rewrite (1.31) as:

$$\begin{aligned}&\sum_K \int_K \varepsilon(v) : C : \varepsilon(u) \\ &-\sum_{f\in\Gamma_*}\int_f ([\![v]\!] : \{\sigma(u)\} + [\![u]\!] : \{\sigma(v)\} - \eta[\![v]\!] : [\![u]\!]) = \sum_K \int_K f\cdot v\end{aligned} \tag{1.33}$$

In the current work, we are interested only in the interelement behavior. We choose to strongly enforce boundary conditions on the external surfaces of the body in the usual manner (rather than weakly prescribing them through the surface integrals via the jump and average operators). Therefore, we can limit the surface integrals to just the set of interior surfaces, $f = \Gamma \in \Gamma_I$:

$$\begin{aligned}&\sum_K \int_K \varepsilon(v) : C : \varepsilon(u) \\ &-\sum_{\Gamma\in\Gamma_I}\int_\Gamma ([\![v]\!] : \{\sigma(u)\} + [\![u]\!] : \{\sigma(v)\} - \eta[\![v]\!] : [\![u]\!]) = \sum_K \int_K f\cdot v\end{aligned} \tag{1.34}$$

Rearranging the interface terms for clarity, we arrive at the general form of the IP DGM:

$$\begin{aligned}&\sum_K \int_K \varepsilon(v) : C : \varepsilon(u) \\ &-\sum_{\Gamma\in\Gamma_I}\left(\int_\Gamma [\![v]\!] : \{\sigma(u)\} + \int_\Gamma [\![u]\!] : \{\sigma(v)\} - \int_\Gamma \eta[\![v]\!] : [\![u]\!]\right) = \sum_K \int_K f\cdot v\end{aligned} \tag{1.35}$$

In addition to the standard (continuous Galerkin) FEM terms, the IP method contains three interface integrals for each $\Gamma \in \Gamma_I$. The first integral ensures consistency of the IP formulation, and is referred to as the *consistency* term. When the consistency term is combined with the second integral, the form is symmetric in u and v. Therefore, the second integral is referred to as the *symmetry* term. The third integral weakly enforces continuity of the displacements and is referred to as the *stability* term.

By expanding each term in (1.36) and using the average and jump operators defined in (1.3) and (1.4), we see that the two surface normals n_1 and n_2 have been preserved, as desired:

$$\begin{aligned}
&\sum_K \int_K \varepsilon(v) : C : \varepsilon(u) \\
&- \sum_{\Gamma \in \Gamma_I} \int_\Gamma (v_1 \otimes n_1 + v_2 \otimes n_2) : \frac{1}{2}\big(\sigma_1(u) + \sigma_2(u)\big) \\
&- \sum_{\Gamma \in \Gamma_I} \int_\Gamma (u_1 \otimes n_1 + u_2 \otimes n_2) : \frac{1}{2}\big(\sigma_1(v) + \sigma_2(v)\big) \\
&+ \sum_{\Gamma \in \Gamma_I} \int_\Gamma \eta\, (v_1 \otimes n_1 + v_2 \otimes n_2) : (u_1 \otimes n_1 + u_2 \otimes n_2) \\
&= \sum_K \int_K f \cdot v
\end{aligned} \tag{1.36}$$

Thus, the IP DGM primal formulation is complete.

Specialization: Weighted Averaging

In composites and adhesive joints, the material properties on either side of an interface may be significantly different. To account for this in the stress flux across the interface, we can use a weighted average operator, which directly affects the consistency and symmetry terms in the IP DGM. For example:

$$\{\sigma\} = \gamma_1\sigma_1 + \gamma_2\sigma_2 \quad \text{where} \quad \gamma_1 + \gamma_2 = 1 \tag{1.37}$$

As shown by Annavarapu [11], the weight factors, γ_i, for each side of an interface may be based on element moduli κ and geometric factors such as the characteristic element size h. A general expression for the weights is then given by:

$$\gamma_i = \frac{h_i/\kappa_i}{h_1/\kappa_1 + h_2/\kappa_2} \quad \text{where } i = 1,2 \tag{1.38}$$

Since the interface weights must sum to unity, (1.37) can be written with the equivalent expression $\{\sigma\} = \gamma_1\sigma_1 + (1-\gamma_1)\sigma_2$ or even $\{\sigma\} = (1-\gamma_2)\sigma_1 + \gamma_2\sigma_2$ if desired. Note that if the material and element size are the same for both element, the weight factors for each side of the interface become $\gamma_i = 1/2$, in which case the weighted average operator is equal to the mean and matches the definition in (1.3).

For our purposes, we calculate the element moduli factor κ_i for each side of the interface with the (Frobenius) induced matrix norm of its constitutive matrix, C:

$$\kappa = \|C\|_{\mathrm{F}} = \sqrt{\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\left|c_{ij}\right|^2\right)} \tag{1.39}$$

Finally, as suggested [11], we use the same concept to consistently scale the user-defined scalar penalty parameter (η_0) used in the IP method stability term, so that it is also updated to account for the constitutive properties and element sizes, and the newly computed interface weights (γ_1) and (γ_2):

$$\eta = \frac{\eta_0\, h_\Gamma}{\frac{h_1}{\kappa_1} + \frac{h_2}{\kappa_2}} \tag{1.40}$$

Specialization: Acoustic Tensor

Importantly, the stability of the IP DGM is influenced by a penalty parameter. As previously mentioned, this parameter must only be sufficiently large to result in a positive definite interface stiffness matrix, unlike the penalty-stiffness used in the conventional CZM. We feel that a global value for all interfaces should be assumed only if there is no other option, and that a minimum penalty parameter may be found by considering each interface independently. Fortunately, in the formulation developed for this report, we have implemented an approach where the penalty is automatically adjusted according to the local conditions at each interface, with very good results.

In the derivations above, the IP DGM stress flux in (1.19), ($\hat{\sigma} = \{C : \varepsilon(u)\} - \eta[\![u]\!]$), is defined to include the penalty parameter η, which is typically assumed to be scalar value. After writing the IP method in its primal form (1.35), the penalty is applied to the stability term and is written:

$$\int_{\Gamma} \eta \, [\![v]\!] : [\![u]\!] \tag{1.41}$$

We choose to allow the penalty to vary for each interface, Γ. In addition, rather than assuming η takes a scalar value, we specialize the penalty parameter to be of the form:

$$\eta = \eta_0 \alpha \tag{1.42}$$

In this expression, η_0 is assumed to be a scalar multiplier such that $\eta_0 \geq 1$, and α represents an "interfacial acoustic tensor", which accounts for constitutive properties, C, for the materials on either side of each interface. For instance, the acoustic tensor of each side of an interface can be given by:

$$\alpha_i = n_i \cdot C_i \cdot n_i^T \quad \text{where } i = 1,2 \tag{1.43}$$

Thus, for a 2D model, the acoustic tensor for each side of an interface takes the form of a 2x2 matrix so that constitutive properties are projected in the direction of the element's outward-facing surface normal, n_i. Therefore, the appropriate form of (1.41) is rewritten:

$$\int_{\Gamma} [\![v]\!] : \eta : [\![u]\!] \tag{1.44}$$

To combine the acoustic tensors from each side of the interface, we assume they can be added like two springs in series, such that:

$$\alpha = 2 \, (\alpha_1^{-1} + \alpha_2^{-1})^{-1} \tag{1.45}$$

Note that if the constitutive properties on either side of the interface are equal, then:

$$\alpha_1 = \alpha_2 = \alpha \tag{1.46}$$

With this definition, the penalty parameter η is intended to automatically balance the contributions from either side of the interface while its components remain of the same order as the material constitutive properties. Thus, the effect of penalization on the condition number of the Global Stiffness Matrix is minimized, while ensuring stability of the IP DGM. Finally, we permit the user to "over-penalize" the IP method stability term by defining the scalar multiplier η_0 in the Abaqus input file. This capability may be useful if the acoustic tensor approach fails to be sufficient to stabilize the method.

--------- Excerpt ends ---------

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