Variations on a Viscoplastic Material Model: Strain Hardening and Strain-Rate Sensitivity

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In this report good use is made of simple classic viscoelastic material models in order to build more complex models, while generalizing for nonlinearities due to strain hardening, strain-rate sensitivity, and plasticity. It is shown that the classical equations may be rewritten in a more convenient form where stress is expressed as a function of strain, making computation more straight-forward for displacement-based load histories. The real-world use of a stress power law strain hardening model for an epoxy-based adhesive material is then demonstrated.

I. Introduction

Fitting a material model to experimental data may be a challenge for cases when strain-hardening and rate-dependency are observed. A solid understanding of the type of behaviors that can be modeled using a variety of classic viscoelastic models may be valuable in such exercises. Using simple springs and dashpots as mechanical analogs, we can build increasingly complex material models. In the following paper, we start with a relatively complex model and present a few variations – often making the model simpler in the process. In this way, the discussion here is given in a somewhat opposite direction to the route taken in the usual textbook approach. It is hoped that this alternative viewpoint provides some insight to other researchers.

We also note that many classic viscoelastic models are formulated so that they are best suited for force-based load histories, and where the stress in the material is known. In practice, such forms are not always convenient. For example, material testing is often performed in the laboratory using displacement-based control – i.e., loading is controlled by actuator displacements. Displacement-control is also typically used when performing fracture tests, in an attempt to obtain stable fracture propagation. Therefore, we investigate techniques to re-express the classic models in a more practical form for applications where the displacement (or strain) history of a material is known. The rewritten forms are also useful for implementing viscoelastic constitutive behavior in the (displacement-based) finite element method.

II. Model Formulation

We take as our starting point the 1-dimensional mechanical analog shown in Fig. 1, which consists of a parallel combination of the familiar Kelvin viscoelastic element with a Saint-Venant dry-friction element, which are then placed in series with a linear spring. We denote this configuration the Linear Elastic Viscoplastic Model, or "LEVM". We will, at first, follow the derivation provided in the excellent work of Shames and Cozzarelli [1] (albeit presented in pieces found in several places within their text). Additional clarifications are provided here whenever needed.

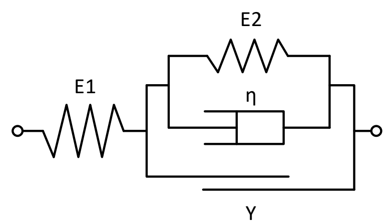

Figure 1. Linear Elastic, Viscoplastic Model ("LEVM")

The constitutive behavior of the linear Kelvin element is expressed by:

$$\tau = \tau_S + \tau_D \qquad \text{Eq. (1)}$$

$$\varepsilon = \varepsilon_S = \varepsilon_D \qquad \text{Eq. (2)}$$

$$\varepsilon_S = \frac{\tau_S}{E} \qquad and \qquad \dot{\varepsilon}_D = \frac{\tau_D}{\eta} \qquad \text{Eq. (3)}$$

$$\tau = \eta\dot{\varepsilon} + E\varepsilon \qquad \text{Eq. (4)}$$

where the subscripts S and D indicate the spring or dashpot, respectively, and the stress and strain are assumed to be functions of time. Following [1], we then rewrite Eq. (4) by introducing the strain retardation time, $t_\varepsilon = \eta/E$, to obtain the alternative form:

$$\dot{\varepsilon} + \frac{\varepsilon}{t_\varepsilon} = \frac{1}{t_\varepsilon}\left(\frac{\tau}{E}\right) \qquad \text{Eq. (5)}$$

The Kelvin element constitutive model can easily be made nonlinear by generalizing Eq. (5) into a stress power-law form. This can be useful to account for strain-hardening behavior:

$$\dot{\varepsilon} + \frac{\varepsilon}{t_\varepsilon} = \frac{1}{t_\varepsilon}\left(\frac{\tau}{\mu_p}\right)^n, \qquad n \geq 1 \qquad \text{Eq. (6)}$$

By comparing Eq. (5) and (6), note that the parameter $\mu_p = E$ when the exponent n=1.

The LEVM also includes a Saint-Venant dry-friction element, which is perfectly rigid for $\tau < Y$ and perfectly plastic for $\tau \geq Y$. Thus, the flow rule for the Saint-Venant element is written:

$$\begin{aligned} \dot{\varepsilon}_p &= 0, \qquad for\,\tau < Y \\ \dot{\varepsilon}_p &\geq 0, \qquad for\,\tau \geq Y \end{aligned} \qquad \text{Eq. (7)}$$

In the LEVM, the Saint-Venant element is placed in parallel with the Kelvin element. Therefore, the strain in each must be equal, and the sum of the stresses in each element must be equal the total applied load. Then, for the Saint-Venant/Kelvin combination we have:

$$\tau = \tau_K + \tau_V \qquad \text{Eq. (8)}$$

$$\varepsilon_p = \varepsilon_K = \varepsilon_V \qquad \text{Eq. (9)}$$

So: $$if\ \tau < Y: \quad \dot{\varepsilon}_p = \dot{\varepsilon}_K = \dot{\varepsilon}_V = 0, \quad \tau_V = \tau, \quad \tau_K = 0 \qquad \text{Eq. (10)}$$

$$if\ \tau \geq Y: \quad \dot{\varepsilon}_p = \dot{\varepsilon}_K = \dot{\varepsilon}_V \geq 0, \quad \tau_V = Y, \quad \tau_K = \tau - Y \qquad \text{Eq. (11)}$$

Note that the strain rate of the plastic part, $\dot{\varepsilon}_p$, is non-zero only for stresses that exceed the yield strength, Y. When $\tau \geq Y$, the plastic strain rate can be found using Eq. (11). Mathematically, this is written:

$$if\ \tau < Y: \quad \dot{\varepsilon}_p = 0 \qquad \text{Eq. (12)}$$

$$if\ \tau \geq Y: \quad \dot{\varepsilon}_p = \frac{1}{t_\varepsilon}\left(\frac{\tau - Y}{\mu_p}\right)^n - \frac{\varepsilon_p}{t_\varepsilon} \qquad \text{Eq. (13)}$$

Next, we note that the LEVM is arranged so that the "plastic" part and the "elastic" part are in series. Therefore, the stress in each part must be equal, and the total strain in the model is the sum from each part:

$$\varepsilon = \varepsilon_e + \varepsilon_p = \frac{\tau}{E} + \varepsilon_p \qquad \text{Eq. (14)}$$

And so:

$$\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_p = \frac{\dot{\tau}}{E} + \dot{\varepsilon}_p \qquad \text{Eq. (15)}$$

Therefore, the LEVM can finally be expressed as:

$$if\ \tau < Y: \quad \dot{\varepsilon} = \frac{\dot{\tau}}{E} \qquad \text{Eq. (16)}$$

$$if\ \tau \geq Y: \quad \dot{\varepsilon} = \frac{\dot{\tau}}{E} + \frac{1}{t_\varepsilon}\left[\left(\frac{\tau - Y}{\mu_p}\right)^n - \varepsilon + \frac{\tau}{E}\right] \qquad \text{Eq. (17)}$$

Taking stock of these developments, we see that the LEVM is a rate-sensitive viscoplastic model that has been generalized to also provide the ability to account for various experimentally observed strain hardening behavior by using the stress power-law form.

For an example application of the LEVM, consider the case where an applied stress is given by the linear form in Eq. (18):

$$\tau(t) = \frac{\tau_0}{t_0} \cdot t \qquad \text{Eq. (18)}$$

so that the stress is constantly increasing at the rate τ_0/t_0. To obtain the stress-strain curve, Eqs. (16) and (17) must be integrated in time. This process is demonstrated in [1] for a somewhat different function of stress, but we have verified the outcome is the same:

$$\varepsilon(\tau) = \frac{\tau}{E} + \frac{\tau - Y}{\mu_p} - \frac{t_\varepsilon \tau_0}{t_0 \mu_p}\left\{1 - \exp\left[-\frac{(\tau - Y)t_0}{t_\varepsilon \tau_0}\right]\right\} \qquad \text{Eq. (19)}$$

In Fig. 2, the LEVM stress-strain curve is shown under three load rates. (To calculate this stress-strain curve, we chose the material properties: Y=40MPa, E=4000MPa, t_ε=80, μ_p=250MPa, $\eta 1$=20000, n=1.)

The stress load-rates shown are: 1 MPa/sec, 2MPa/sec, and 10MPa/sec. As illustrated by the figure, the material exhibits the onset of nonlinearity and plasticity at the yield stress, after which the stress-strain curve "lifts" (becomes more stiff) for increasing load rates.

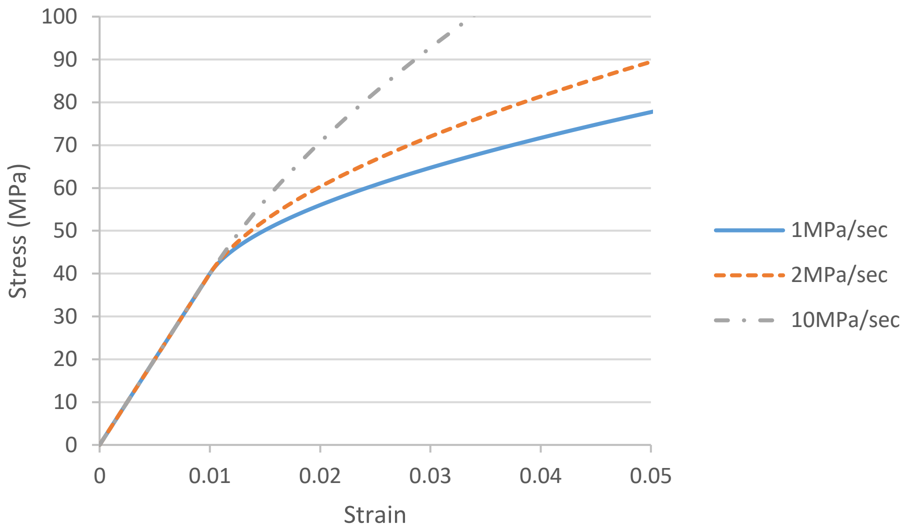

Figure 2. Linear Elastic Viscoplastic Model with (LEVM) with load rate sensitivity in the plastic region.

III. Variations on the Viscoplastic Model

We next show that a few variations of the LEVM can be used to represent different material behaviors, such as rate-independence, plasticity, strain hardening, solid-like rate-dependent behavior, and fluid-like rate-dependent behavior.

A Rate-Independent Model with Plasticity and Strain Hardening:

We begin with a useful variation of the LEVM that can be used to represent a rate-independent Linear Elastic Plastic Model (or "LEPM"). We begin by multiplying Eq. (17) by t_ε and then setting $t_\varepsilon = \eta/E = 0$, to obtain:

$$if\ \tau < Y: \quad \varepsilon = \frac{\tau}{E} \qquad \text{Eq. (20)}$$

$$if\ \tau \geq Y: \quad \varepsilon = \frac{\tau}{E} + \left(\frac{\tau}{\mu_p}\right)^n \qquad \text{Eq. (21)}$$

The rate-independent power-law form of Eq. (21) is easily invertible for $n = 1$ or 2, but becomes more difficult as $n \geq 3$.

$$n = 1: \qquad \tau = \frac{E\mu_p\varepsilon + EY}{\mu_p + E} \qquad \text{Eq. (22)}$$

$$\tau = \left[\frac{\varepsilon}{A} - \frac{C}{A} + \left(\frac{B}{2A}\right)^2\right]^{\frac{1}{2}} - \frac{B}{2A}$$

$n = 2$: Eq. (23)

$$with \quad A = \frac{1}{\mu_p^2}, \quad B = \frac{1}{E} - \frac{2Y}{\mu_p^2}, \quad C = \frac{Y^2}{\mu_p^2}$$

Fig. 3 illustrates the stress-strain curves of the LEPM for n = [1, 2] and Y=40MPa, E=4000MPa, μ_p=250MPa.

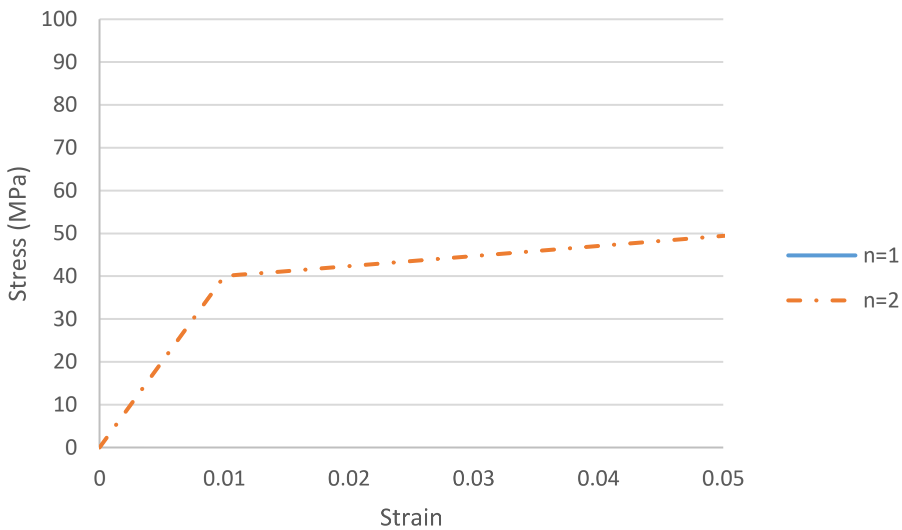

Figure 3: Linear Elastic Plastic Model with Stress Power Law Strain Hardening (LEPM).

A Solid-Like Viscoelastic Model:

A second variation is found by removing the Saint-Venant element from the LEVM. We then obtain the 3-Parameter Standard Solid (Fig. 4). Notice that in this model, plastic behavior cannot be modeled. As before, we see from a balance of forces that each element in this configuration has the same stress, while the strains in each element are different.

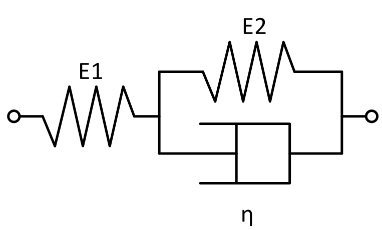

Figure 4. Standard 3-Parameter Solid

The total strain is the sum over each element:

$$\varepsilon = \varepsilon_1 + \varepsilon_2 \qquad \text{Eq. (24)}$$

We can algebraically solve for the strain over the Kelvin element using Eq. (4) and the D-derivative operator:

$$\tau = \eta D\varepsilon_2 + E_2\varepsilon_2 \qquad \text{Eq. (25)}$$

Or: $$\varepsilon_2 = \frac{\tau}{E_2 + \eta D}$$

Then Eq. (24) becomes: $$\varepsilon = \frac{\tau}{E_1} + \frac{\tau}{E_2 + \eta D} = \frac{E_2 + \eta D + E_1}{E_1E_2 + E_1\eta D} \cdot \tau$$

Rewriting: $$E_1E_2\varepsilon + E_1\eta\dot{\varepsilon} = (E_1 + E_2)\tau + \eta\dot{\tau}$$

Or, finally: $$\tau + \frac{\eta}{E_1 + E_2}\dot{\tau} = \frac{E_1E_2}{E_1 + E_2}\varepsilon + \frac{E_1\eta}{E_1 + E_2}\dot{\varepsilon} \qquad \text{Eq. (26)}$$

IV. Re-expressing the Models as a Function of Strain Only

We note many of the formulations provided in classical viscoelasticity are more convenient when the strain is a function of a known applied load, i.e., $\varepsilon = \varepsilon(\tau)$. Aside from creep and compliance tests, this is not always very convenient. However, we can derive expressions in the inverse form, i.e. $\tau = \tau(\varepsilon)$. This is easily done for the simpler mechanical analogs by straightforward algebraic manipulation, but for more complex models a different approach may be required.

To illustrate the process, we first rewrite the viscoelastic equations in a generalized form (following Eqs. 27-29). For example, the differential equation for standard 3-Parameter Solid from Eq. (26) can be rewritten as in Eq. (30).

Generalized Viscoelastic Diff. Eqn. $$p_0\tau + p_1\dot{\tau} + p_2\ddot{\tau} + \cdots p_m\tau^{(m)} = q_0\varepsilon + q_1\dot{\varepsilon} + q_2\ddot{\varepsilon} + \cdots + q_m\varepsilon^{(m)} \qquad \text{Eq. (27)}$$

$$P = \sum_{j=0}^{m} p_j \frac{\partial^j}{\partial t^j} = \sum_{j=0}^{m} p_j D^j \qquad \text{Eq. (28)}$$

$$Q = \sum_{j=0}^{m} q_j \frac{\partial^j}{\partial t^j} = \sum_{j=0}^{m} q_j D^j \qquad \text{Eq. (29)}$$

$$P\tau = Q\varepsilon \qquad \text{Eq. (30)}$$

Now, to facilitate the algebraic manipulation of the viscoelastic equation to a more convenient form, we take the Laplace Transform:

$$\bar{P}(s)\bar{\tau}(s) = \bar{Q}(s)\bar{\varepsilon}(s) \qquad \text{Eq. (31)}$$

Rewriting: $$\bar{\tau}(s) = \frac{\bar{Q}(s)}{\bar{P}(s)}\bar{\varepsilon}(s)$$

$$= \frac{E_1E_2 + E_1\eta s}{E_1 + E_2 + \eta s}\bar{\varepsilon}(s) \qquad \text{Eq. (32)}$$

Now, assuming the applied strain is given by a linear function of t:

$$\varepsilon(t) = \frac{\varepsilon_0}{t_0}\cdot t \qquad \text{Eq. (33)}$$

We have: $$\bar{\varepsilon}(s) = \frac{\varepsilon_0}{t_0}\frac{1}{s^2} \qquad \text{Eq. (34)}$$

So: $$\bar{\tau}(s) = \frac{E_1E_2 + E_1\eta s}{E_1 + E_2 + \eta s}\frac{\varepsilon_0}{t_0}\frac{1}{s^2} \qquad \text{Eq. (35)}$$

Taking the Inverse Laplace of Eq. (35) we then obtain:

$$\tau(t) = \frac{\varepsilon_0 E_1}{t_0(E_1+E_2)^2}\left(E_2^2\cdot t + \eta E_1 + E_1E_2\cdot t - \eta E_1 \exp\left[-\left(\frac{E_1+E_2}{\eta}\right)\cdot t\right]\right) \qquad \text{Eq. (36)}$$

In Eq. (36) the stress is given as a function of time, as well as the strain rate. However, we can remove an explicit dependence upon t if the applied strain-based loading rate is known. For example, if the applied strain is given by Eq. (33), we have the strain-rate:

$$\dot{\varepsilon}(t) = \frac{\varepsilon_0}{t_0} \qquad \text{Eq. (37)}$$

Therefore, the stress can be expressed as:

$$\tau(\varepsilon) = \frac{\varepsilon_0 E_1}{t_0(E_1+E_2)^2}\left(\frac{\varepsilon(t)t_0}{\varepsilon_0}E_2^2 + \eta E_1 + \frac{\varepsilon(t)t_0}{\varepsilon_0}E_1E_2 - \eta E_1 \exp\left[-\frac{\varepsilon(t)t_0}{\varepsilon_0}\left(\frac{E_1+E_2}{\eta}\right)\right]\right) \qquad \text{Eq. (38)}$$

The form of Eq. (38) is rather complex, but is capable of capturing some interesting viscoelastic behavior. For example, see Fig. 5 for the 3-Parameter Standard Solid, with $\tau = \tau(\varepsilon)$ calculated for three different strain rates, and where we have chosen the material parameters as: E_1=4000MPa, E_2=2000MPa, and η=50000.

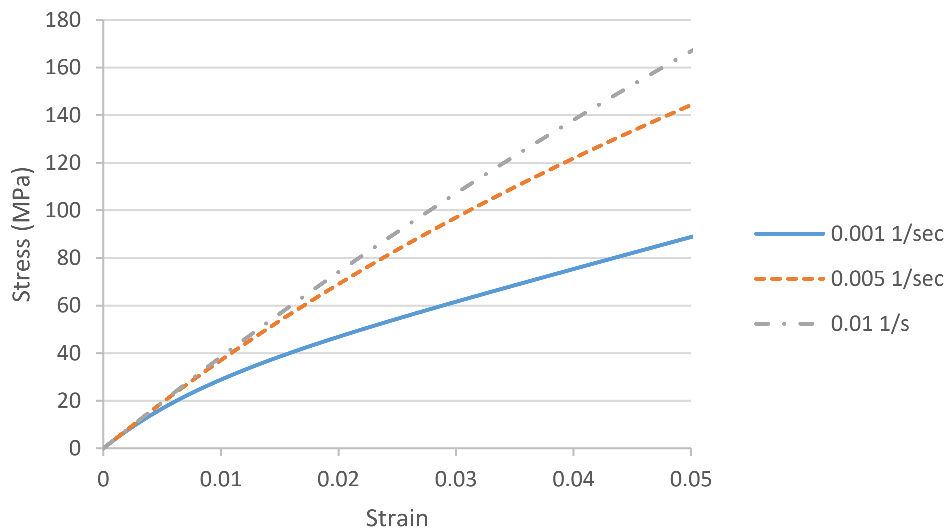

Figure 5. 3-Parameter Standard Solid, for τ=τ(ε), for 3 constant strain rates.

Note that for this solid-like model, the stress will monotonically increase for increasing levels of strain. We will next investigate a material model in which this does not occur.

A Fluid-Like Viscoelastic Model:
Another popular viscoelastic (rate-dependent) model is the Burger's Fluid, illustrated in Fig. 5, which consists of the familiar Maxwell Fluid and Kelvin Solid models in series. Following the same procedure as described above for the 3-Parameter Standard Solid, we will begin by noting that the total strain across the model is equal to the sum of the strains in each element. Again using the *D* operator, we can express the Maxwell and Kelvin models as:

$$\varepsilon_M = \frac{\tau}{E_1} + \frac{\tau}{\eta_1 D}$$

$$\varepsilon_K = \frac{\tau}{E_2 + \eta_2 D}$$

Therefore:
$$\varepsilon = \varepsilon_m + \varepsilon_K = \frac{\tau}{E_1} + \frac{\tau}{\eta_1 D} + \frac{\tau}{E_2 + \eta_2 D} \qquad \text{Eq. (39)}$$

Rewriting:
$$\eta_1 \dot{\varepsilon} + \frac{\eta_1 \eta_2}{E_2} \ddot{\varepsilon} = \tau + \frac{E_2 \eta_1 + E_1 \eta_2 + E_1 \eta_1}{E_1 E_2} \dot{\tau} + \frac{\eta_1 \eta_2}{E_1 E_2} \ddot{\tau} \qquad \text{Eq. (40)}$$

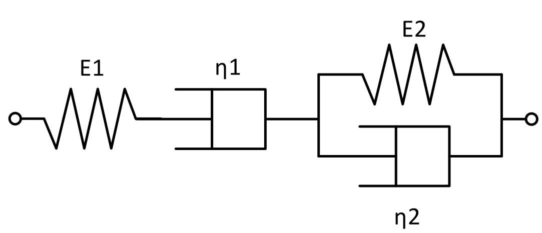

Figure 6. Burger's Fluid Model

Following the process described above for the 3-Parameter Solid, we express Eq. (40) in the form of the polynomial operators Q and P. This permits us to algebraically solve the equation in the Laplace transform space so that stress is expressed as a function of strain and time, $\tau = \tau(\varepsilon, t)$. Then, by taking the inverse Laplace transform and removing the explicit dependence upon time by substituting a known strain function, we arrive at the final equation of the Burger's Fluid model in the form of $\tau = \tau(\varepsilon)$ given in Eq. (41).

$$\tau(\varepsilon) = \left(\frac{\varepsilon_0}{t_0}\right)\left(\frac{\eta_1}{E_2}\right)(E_2 + \eta_2) \cdot \left\{1 - \exp\left[-\left(\frac{E_1 E_2}{E_1\eta_1 + E_1\eta_2 + E_2\eta_1 + \eta_1\eta_2}\right)\left(\varepsilon(t) \cdot \frac{t_0}{\varepsilon_0}\right)\right]\right\} \quad \text{Eq. (41)}$$

The results for three different strain rates are illustrated in Fig. 7 for the material parameters: E_1=3400MPa, E_2=250MPa, η_1=50, η_2=20. Notice that for this fluid-like model, the stress eventually plateaus. This is unlike the solid-like models described above, where the stress continues to increase for increasing levels of strain.

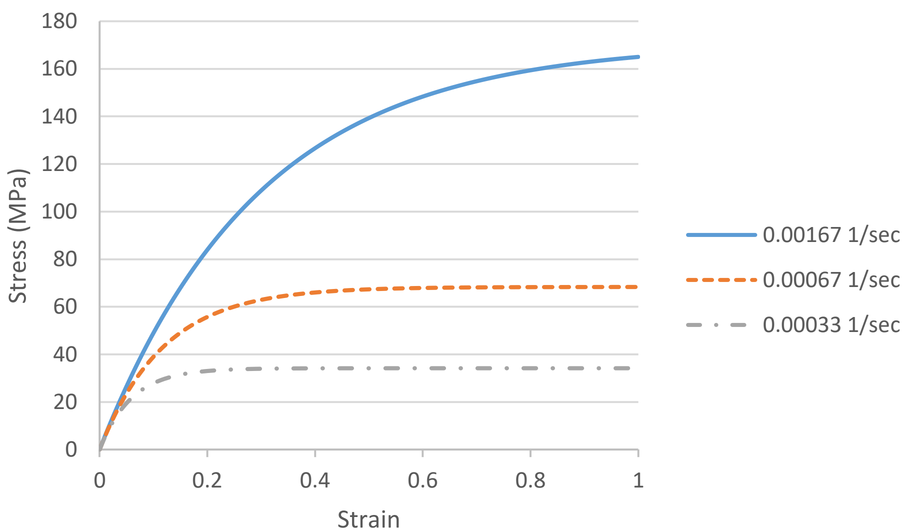

Figure 7. Rate Dependent Burger's Fluid, for τ=τ(ε), at three different constant strain rates.

V. An Example Application

An epoxy-based adhesive used in wind turbine blade manufacturing has undergone mechanical testing at the Montana State University Composites Group. Typical quasi-static stress-strain curves are shown in Fig. 8 for a "neat" tensile specimen (black line) and a neat compression test specimen (red line). Note that the elastic modulus (E) and yield stress (Y) can immediately be determined from the experimental stress-strain curve. The rate-independent LEPM matches this behavior very closely, where we have used Eq. (23) with the 4 parameters chosen as (E=4000MPa, Y=40MPa, μ_p=250MPa, $n=2$).

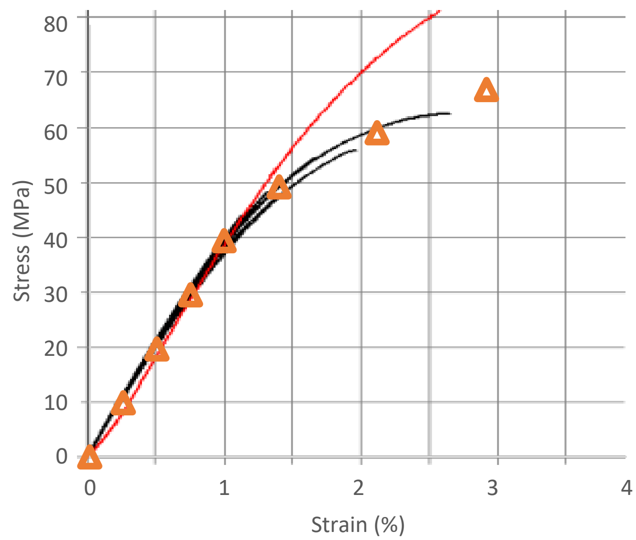

Figure 8. Stress Strain curve for an epoxy-based adhesive.
Note the LEPM (*n*=2) matches the tensile test data closely.

A version of this material behavior is currently under development for use as a constitutive model in a finite element code to represent damaged adhesive materials.

VI. Summary

In this work, we derived several viscoelastic and viscoplastic models and discussed their general stress-strain behavior. Computationally convenient forms were then found such that $\tau = \tau(\varepsilon)$, making their use more suitable for the common situation where loads are displacement-based and only the strain history of the material is known. An example real-world application for these models was then demonstrated. Understanding the classic forms may be extremely useful for representing nonlinear material behavior.

References

[1] Shames, Irving H., Cozzarelli, Francis A., “Elastic and Inelastic Stress Analysis”, Revised Printing, 1997. Section 8.3.